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SIGNALING PROBLEM FOR TIME-FRACTIONAL DIFFUSION-WAVE EQUATION IN A HALF-PLANE

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Abstract

The time-fractional diffusion-wave equation is considered in a half-plane. The Caputo fractional derivative of the order $0 < \alpha < 2$ is used. Several examples of problems with Dirichlet and Neumann boundary conditions are solved using the Laplace integral transform with respect to time and the Fourier transforms with respect to spatial coordinates. The solution is written in terms of the Mittag-Leffler functions. For the first and second time-derivative terms, the obtained solutions reduce to the solutions of the ordinary diffusion and wave equations. Numerical results are illustrated graphically.

Mathematics Subject Classification: 45K05, 26A33, 35J05, 35K05, 35L05
Key Words and Phrases: fractional calculus, diffusion-wave equation, Mittag-Leffler functions

1. Introduction

In recent years considerable interest has been shown in time-fractional diffusion equation

$$\frac{\partial^\alpha c}{\partial t^\alpha} = a\,\Delta c, \qquad 0 < \alpha \leq 2 \tag{1}$$

which describes important physical phenomena in amorphous, colloid, glassy and porous materials, in fractals and percolation clusters, comb structures, dielectric and semiconductors, biological systems, polymers, random and disordered media, geophysical and geological processes (see [1–5] and references therein).

Two types of anomalous transport can be distinguished. The slow diffusion is exemplified by the value $0 < \alpha < 1$, whereas the fast diffusion (fractional wave propagation) is characterized by the value $1 < \alpha < 2$. The limiting cases $\alpha = 0$ and $\alpha = 2$ are known as localized and ballistic diffusion and correspond to the Helmholtz and wave equations, respectively.

Essentials of fractional calculus can be found in the treatises [6–11] and the reviews [12, 13]. Several definitions of a fractional derivative have been proposed. Samko, Kilbas and Marichev [7] provided an excellent historical review of this subject. For its Laplace transform rule the Riemann-Liouville derivative of the fractional order requires the knowledge of the initial values of the fractional integral of the function and its derivatives, while the Caputo fractional derivative needs the initial values of the function and its integer derivatives. Formulae establishing relations between these two types of fractional derivatives can be found in [12, 14]. It should be emphasized that if care is taken, the results obtained using the Caputo formulation can be recast to the Riemann-Liouville version.

The fundamental solution for the fractional diffusion-wave equation in one space-dimension was obtained by Mainardi [15] who also considered the signaling problem and the evolution of the initial box-signal [16] using the Laplace transform. Wyss [17] obtained the solution of the Cauchy and signaling problems in terms of H-functions using the Mellin transform. Schneider and Wyss [18] converted the diffusion-wave equation with appropriate initial conditions into the integrodifferential equation and found the corresponding Green functions in terms of Fox functions. Metzler and Klafter [19] considered the fractional diffusion equation on a half-line and a segment with reflecting (the Neumann problem) or absorbing (the Dirichlet problem) boundary conditions. Fujita [20] treated integrodifferential equation which interpolates the diffusion equation and the wave equation and exhibit properties peculiar to both these equations. Hilfer [21] presented a solution of fractional diffusion equation based on Riemann-Liouville fractional derivative in terms of H-functions using the Fourier, Laplace and Mellin transforms. Hanyga [22] studied Green's functions and propagator functions in one, two and three dimensions. Kilbas, Trujillo and Voroshilov [23] studied the Cauchy problem for diffusion-wave equation with Riemann-Liouville time-fractional derivative in R^m. Agrawal [24–27] analized the fractional diffusion-wave equation and the fourth-order fractional diffusion-wave equation both in a half-line and a bounded domain $(0, L)$. Gorenflo and Mainardi [28] studied time-fractional, spatially one-dimensional diffusion-wave equation on the spatial half-line with zero initial conditions. They considered the Dirichlet and Neumann boundary conditions and proved that the Dirichlet-Neumann map is given by a time-fractional differential operator whose order is half the order of the time-fractional derivative. Mainardi and Paradisi [29] used the time-fractional diffusion-wave equation to study the propagation of stress waves in viscoelastic media relevant to acoustics and seismology. In [30] the spatially two-dimensional time-fractional wave equation was used to simulate constant-Q wave propagation. A plane wave was considered, fractional derivatives were computed with the Grünwald-Letnikov and central-difference approximations, and the phase velocity corresponding to each Fourier component was obtained. Several initial and boundary-value problems for time-fractional diffusion-wave equation were considered by the author [31–35].

In the present paper, we study the solutions of time-fractional diffusion-wave equation in a half-plane. Several examples of problems with Dirichlet and Neumann boundary conditions are solved using the Laplace integral transform with respect to time and the Fourier transforms with respect to spatial coordinates. The inverse Laplace transform is expressed in terms of the Mittag-Leffler functions. After inversion of Fourier transform the solution is converted into a form amenable to numerical treatment. For the first and second time-derivative terms, the obtained solutions reduce to the solutions of the ordinary diffusion and wave equations.

2. Statement of the problem

In this paper, we study the time-fractional diffusion equation in a half-plane

$$\frac{\partial^\alpha c}{\partial t^\alpha} = a\left(\frac{\partial^2 c}{\partial x^2} + \frac{\partial^2 c}{\partial y^2}\right), \quad \begin{array}{l} 0 < x < \infty, \quad -\infty < y < \infty, \\ 0 < t < \infty, \quad 0 < \alpha \le 2 \end{array} \tag{2}$$

under zero initial conditions

$$t = 0: \quad c = 0, \qquad 0 < \alpha \le 2, \tag{3}$$

$$t = 0: \quad \frac{\partial c}{\partial t} = 0, \qquad 1 < \alpha \le 2. \tag{4}$$

Two types of boundary conditions at the surface $x = 0$ are discussed: the Dirichlet boundary condition with the prescribed boundary value of the sought-for function

$$x = 0: \quad c = u(y, t) \tag{5}$$

and the Neumann boundary condition with the prescribed boundary value of the normal derivative

$$x = 0: \quad \frac{\partial c}{\partial x} = w(y, t). \tag{6}$$

The zero conditions at infinity are also assumed

$$\lim_{x \to \infty} c(x, y, t) = 0, \qquad \lim_{y \to \pm\infty} c(x, y, t) = 0. \tag{7}$$

In Eq. (2), we use the Caputo fractional derivative

$$\frac{\mathrm{d}^\alpha f(t)}{\mathrm{d}t^\alpha} = \begin{cases} \dfrac{1}{\Gamma(n-\alpha)} \displaystyle\int_0^t (t-\tau)^{n-\alpha-1} \frac{\mathrm{d}^n f(\tau)}{\mathrm{d}\tau^n} \, \mathrm{d}\tau, & n-1 < \alpha < n, \\ \dfrac{\mathrm{d}^n f(t)}{\mathrm{d}t^n}, & \alpha = n. \end{cases} \tag{8}$$

For its Laplace transform rule this derivative requires the knowledge of the initial values of the function and its integer derivatives of order $k = 1, 2, \ldots, n-1$:

$$\mathcal{L}\left\{\frac{\mathrm{d}^\alpha f(t)}{\mathrm{d}t^\alpha}\right\} = s^\alpha \mathcal{L}\{f(t)\} - \sum_{k=0}^{n-1} f^{(k)}(0^+) s^{\alpha-1-k}, \quad n-1 < \alpha < n, \tag{9}$$

where s is the Laplace transform variable.

In the ordinary way, the sin-Fourier transform with respect to the spatial coordinate x will be used in the case of the first boundary value problem with the Dirichlet boundary condition, whereas the cos-Fourier transform will be used in the case of the second boundary value problem with the Neumann boundary condition.

3. The Dirichlet boundary condition

3.1. The fundamental solution

Consider the fundamental solution to the initial-boundary-value problem (2)–(5). In this case we have the following boundary condition

$$x = 0 : \quad c = U_0 \delta(y) \delta_+(t), \tag{10}$$

where $\delta(y)$ is Dirac's delta function, U_0 =const.

Using the Laplace transform with respect to time t, sin-Fourier transform with respect to the spatial coordinate x and exponential Fourier transform with respect to the spatial coordinate y we obtain

$$c^* = \frac{aU_0 \xi}{\sqrt{2\pi}} \frac{1}{s^\alpha + a(\xi^2 + \eta^2)}. \tag{11}$$

Here transforms are denoted by the asterisk, and s, ξ and η are the transform variables.

To invert the Laplace transform the following formula [11, 12] is used

$$\mathcal{L}^{-1} \left\{ \frac{s^{\alpha-\beta}}{s^\alpha + b} \right\} = t^{\beta-1} \, E_{\alpha,\beta}(-bt^\alpha), \tag{12}$$

where $E_{\alpha,\beta}(z)$ is the generalized Mittag-Leffler function in two parameters α and β

$$E_{\alpha,\beta}(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\alpha n + \beta)}, \qquad \alpha > 0, \ \beta > 0, \ z \in C. \tag{13}$$

Inverting the integral transforms we get

$$c = \frac{aU_0 t^{\alpha-1}}{\pi^2} \int_{-\infty}^{\infty} \cos(y\eta) \, \mathrm{d}\eta \int_0^\infty E_{\alpha,\alpha}[-a(\xi^2 + \eta^2)t^\alpha] \, \xi \sin(x\xi) \, \mathrm{d}\xi. \tag{14}$$

Solution (14) is inconvenient for numerical treatment. To obtain the solution amenable for numerical calculations, we pass to polar coordinates in the (ξ, η)-plane and in the (x, y)-plane: $\xi = \rho \cos \vartheta$, $\eta = \rho \sin \vartheta$, $x = r \cos \varphi$, $y = r \sin \varphi$. Then Eq. (14) is rewritten as

$$c = \frac{aU_0 t^{\alpha-1}}{\pi^2} \int_0^\infty \rho^2 E_{\alpha,\alpha}(-a\rho^2 t^\alpha) \, \mathrm{d}\rho \int_{-\pi/2}^{\pi/2} \sin(x\rho \cos \vartheta) \cos(y\rho \sin \vartheta) \cos \vartheta \, \mathrm{d}\vartheta. \tag{15}$$

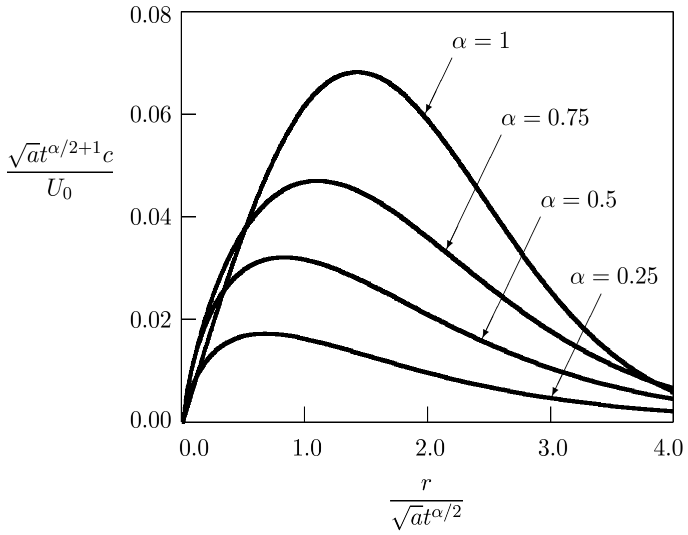

Figure 1: Dependence of solution on distance for $\varphi = 0$ (the delta pulse boundary condition for a function; $0 < \alpha \leq 1$).

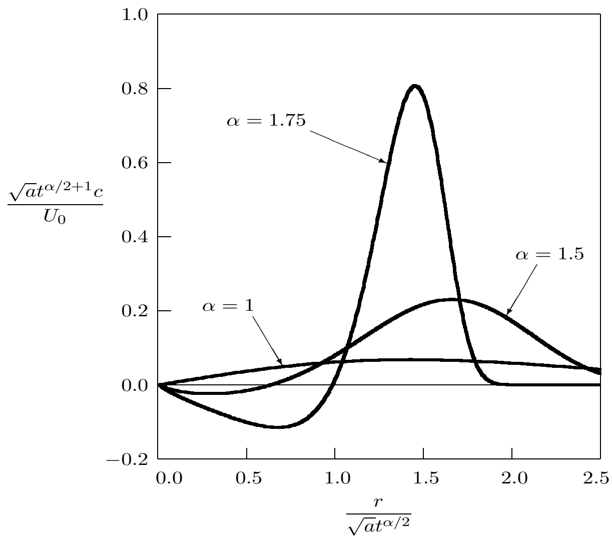

Figure 2: Dependence of solution on distance for $\varphi = 0$ (the delta pulse boundary condition for a function; $1 \leq \alpha < 2$).

Substitution $v = \sin\vartheta$ with taking into account integral (A4) gives

$$\bar{c} = \frac{\cos\varphi}{\pi} \int_0^\infty \sigma^2 E_{\alpha,\alpha}(-\sigma^2)\, J_1(\bar{r}\sigma)\, \mathrm{d}\sigma, \tag{16}$$

where the nondimensional quantities are introduced

$$\bar{c} = \frac{\sqrt{a}t^{\alpha/2+1}}{U_0} c, \quad \bar{r} = \frac{r}{\sqrt{a}t^{\alpha/2}}, \quad \sigma = \rho\sqrt{a}t^{\alpha/2}. \tag{17}$$

Let us analize several particular cases.

3.1.1. Classical diffusion ($\alpha = 1$)

In the case of standard diffusion equation

$$E_{1,1}(-\sigma^2) = \mathrm{e}^{-\sigma^2}, \tag{18}$$

and using (A11) we get (see, for example, [36])

$$\bar{c} = \frac{\bar{r}\cos\varphi}{4\pi} \exp\left(-\frac{\bar{r}^2}{4}\right). \tag{19}$$

3.1.2. Subdiffusion with $\alpha = 1/2$

In this case [33]

$$E_{1/2,1/2}(-\sigma^2) = \frac{2}{\sqrt{\pi}} \int_0^\infty v\, \mathrm{e}^{-v^2-2\sigma^2 v}\, \mathrm{d}v. \tag{20}$$

Inserting (20) into (16), changing integration with respect to σ and v and taking into account (A11) we arrive at

$$\bar{c} = \frac{\bar{r}\cos\varphi}{8\pi^{3/2}} \int_0^\infty \exp\left(-v^2 - \frac{\bar{r}^2}{8v}\right) \frac{1}{v}\, \mathrm{d}v. \tag{21}$$

Dependence of nondimensional function $\bar{c}$ on nondimensional radial coordinate $\bar{r}$ is shown in Figs. 1 and 2 for $\varphi = 0$.

3.2. The constant boundary value of a function in a local domain

Of special interest is the initial-boundary-value problem (2)–(5) with the constant boundary value of a function in the domain $|y| < l$:

$$x = 0: \quad c = \begin{cases} u_0, & |y| < l, \\ 0, & |y| > l. \end{cases} \tag{22}$$

The integral transforms lead to

$$c^* = \frac{2au_0\xi\sin(l\eta)}{\sqrt{2\pi}\eta}\,\frac{1}{s[s^\alpha + a(\xi^2+\eta^2)]}. \tag{23}$$

As

$$\frac{1}{s[s^\alpha + a(\xi^2+\eta^2)]} = \frac{1}{a(\xi^2+\eta^2)}\left[\frac{1}{s} - \frac{s^{\alpha-1}}{s^\alpha + a(\xi^2+\eta^2)}\right] \tag{24}$$

and

$$\mathcal{L}^{-1}\left\{\frac{s^{\alpha-1}}{s^\alpha + a(\xi^2+\eta^2)}\right\} = E_\alpha\left[-a(\xi^2+\eta^2)t^\alpha\right], \tag{25}$$

the solution reads

$$c = \frac{2u_0}{\pi^2}\int_{-\infty}^{\infty}\frac{\sin(l\eta)}{\eta}\,\cos(y\eta)\,\mathrm{d}\eta\int_0^\infty\frac{\xi\sin(x\xi)}{\xi^2+\eta^2}\left\{1 - E_\alpha[-a(\xi^2+\eta^2)t^\alpha]\right\}\,\mathrm{d}\xi. \tag{26}$$

Introducing polar coordinates in the (ξ,η)-plane and taking into account integrals (A1), (A2) and (A6) from Appendix we obtain

$$\bar{c} = \frac{1}{\pi}\left[\arctan\frac{1-\bar{y}}{\bar{x}} + \arctan\frac{1+\bar{y}}{\bar{x}}\right]$$

$$-\frac{\bar{x}}{\pi}\int_0^\infty E_\alpha(-\kappa^2\sigma^2)\,\mathrm{d}\sigma\int_{\bar{y}-1}^{\bar{y}+1}\frac{1}{\sqrt{u^2+\bar{x}^2}}\,J_1\left(\sigma\sqrt{u^2+\bar{x}^2}\right)\,\mathrm{d}u, \tag{27}$$

where the following nondimensional quantities

$$\bar{c} = \frac{c}{u_0},\quad \bar{x} = \frac{x}{l},\quad \bar{y} = \frac{y}{l},\quad \kappa = \frac{\sqrt{a}t^{\alpha/2}}{l} \tag{28}$$

are introduced.

Consider several particular cases.

3.2.1. Classical diffusion ($\alpha = 1$)

The solution has the form (see also [36])

$$\bar{c} = \frac{\bar{x}}{\pi}\int_{\bar{y}-1}^{\bar{y}+1}\frac{1}{u^2+\bar{x}^2}\exp\left(-\frac{u^2+\bar{x}^2}{4\kappa^2}\right)\mathrm{d}u \tag{29}$$

when it is considered that

$$E_1(-\kappa^2\sigma^2) = \mathrm{e}^{-\kappa^2\sigma^2} \tag{30}$$

and Eq. (A9) is used.

3.2.2. Localized diffusion ($\alpha = 0$)

In this case

$$E_0(-\kappa^2\sigma^2) = \frac{1}{1+\kappa^2\sigma^2} \tag{31}$$

and (see also [36])

$$\bar{c} = \frac{\bar{x}}{\pi\kappa}\int_{\bar{y}-1}^{\bar{y}+1} \frac{1}{\sqrt{u^2+\bar{x}^2}}\, K_1\left(\frac{\sqrt{u^2+\bar{x}^2}}{\kappa}\right) \mathrm{d}u, \tag{32}$$

where (A13) and (A14) have been taken into account.

3.2.3. Subdiffusion with $\alpha = 1/2$

The Mittag-Leffler function of the order 1/2 is well-known [37]

$$E_{1/2}(-\kappa^2\sigma^2) = \mathrm{e}^{\kappa^4\sigma^4}\mathrm{erfc}\left(\kappa^2\sigma^2\right) = \frac{2}{\sqrt{\pi}}\,\mathrm{e}^{\kappa^4\sigma^4}\int_{\kappa^2\sigma^2}^{\infty}\mathrm{e}^{-u^2}\mathrm{d}u,$$

where $\mathrm{erfc}\, x$ is the complementary error function. Substitution $v = u - \kappa^2\sigma^2$ allows us to get representation more convenient for subsequent treatment [33]

$$E_{1/2}(-\kappa^2\sigma^2) = \frac{2}{\sqrt{\pi}}\int_0^{\infty}\mathrm{e}^{-v^2-2\kappa^2\sigma^2 v}\mathrm{d}v. \tag{33}$$

Inserting (33) into (27), changing integration with respect to σ and v and taking into account (A9) we obtain

$$\bar{c} = \frac{2\bar{x}}{\pi^{3/2}}\int_{\bar{y}-1}^{\bar{y}+1}\frac{1}{u^2+\bar{x}^2}\,\mathrm{d}u\int_0^{\infty}\exp\left(-v^2-\frac{\bar{x}^2+u^2}{8\kappa^2 v}\right)\mathrm{d}v. \tag{34}$$

3.2.4. Ballistic diffusion ($\alpha = 2$)

In the case of wave equation

$$E_2(-\kappa^2\sigma^2) = \cos(\kappa\sigma). \tag{35}$$

Having regard to (A7), (A8) and (A16), we obtain the solution which analytical form depends on κ. As $\bar{c}$ is an even function of y, we can consider only $y \geq 0$ and obtain:

(i) $0 < \kappa < |1 - \bar{y}|$

$$\bar{c} = \begin{cases} \frac{1}{2}[1 + \operatorname{sign}(1 - \bar{y})], & 0 < \bar{x} < \kappa, \\ 0, & \kappa < \bar{x} < \infty; \end{cases} \tag{36}$$

(ii) $|1 - \bar{y}| < \kappa < 1 + \bar{y}$

$$\bar{c} = \begin{cases} \frac{1}{2} + \frac{1}{\pi}\arctan\frac{\kappa(1-\bar{y})}{\bar{x}\sqrt{\kappa^2-\bar{x}^2-(1-\bar{y})^2}}, & 0<\bar{x}<\sqrt{\kappa^2-(1-\bar{y})^2}, \\ \frac{1}{2}[1 + \operatorname{sign}(1 - \bar{y})], & \sqrt{\kappa^2-(1-\bar{y})^2}<\bar{x}<\kappa, \\ 0, & \kappa<\bar{x}<\infty; \end{cases} \tag{37}$$

(iii) $1 + \bar{y} < \kappa < \infty$

$$\bar{c} = \begin{cases} \frac{1}{\pi}\arctan\frac{\kappa(1-\bar{y})}{\bar{x}\sqrt{\kappa^2-\bar{x}^2-(1-\bar{y})^2}} & \\ +\frac{1}{\pi}\arctan\frac{\kappa(1+\bar{y})}{\bar{x}\sqrt{\kappa^2-\bar{x}^2-(1+\bar{y})^2}}, & 0<\bar{x}<\sqrt{\kappa^2-(1+\bar{y})^2}, \\ \frac{1}{2} + \frac{1}{\pi}\arctan\frac{\kappa(1-\bar{y})}{\bar{x}\sqrt{\kappa^2-\bar{x}^2-(1-\bar{y})^2}}, & \sqrt{\kappa^2-(1+\bar{y})^2}<\bar{x}<\sqrt{\kappa^2-(1-\bar{y})^2}, \\ \frac{1}{2}[1 + \operatorname{sign}(1 - \bar{y})], & \sqrt{\kappa^2-(1-\bar{y})^2}<\bar{x}<\kappa, \\ 0, & \kappa<\bar{x}<\infty. \end{cases} \tag{38}$$

Dependence of nondimensional solution $\bar{c}$ on nondimensional spatial coordinate x/l is shown in Figs. 3-5 for various values of y/l and κ.

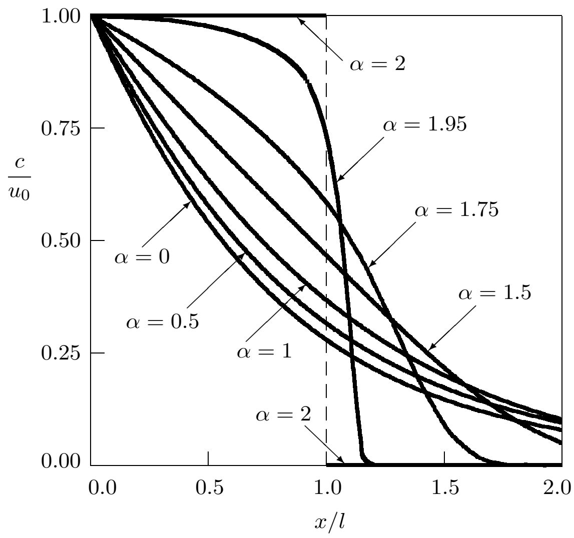

Figure 3: Dependence of solution on distance x/l for $y = 0$ and $\kappa = 1$ (the Dirichlet boundary condition).

4. The Neumann boundary condition

4.1. The fundamental solution

Consider the initial-boundary value problem (2)–(4), (6) with the boundary condition

$$x = 0: \quad \frac{\partial c}{\partial x} = -W_0 \delta(y) \delta_+(t). \tag{39}$$

The Laplace transform with respect to time t, cos-Fourier transform with respect to the spatial coordinate x and exponential Fourier transform with respect to the spatial coordinate y result in

$$c^* = \frac{a W_0}{\sqrt{2\pi}} \frac{1}{s^\alpha + a(\xi^2 + \eta^2)} \tag{40}$$

and after inversion of integral transforms give

$$c = \frac{a W_0 t^{\alpha-1}}{\pi^2} \int_{-\infty}^{\infty} \cos(y\eta)\, \mathrm{d}\eta \int_0^\infty E_{\alpha,\alpha}[-a(\xi^2 + \eta^2) t^\alpha] \cos(x\xi)\, \mathrm{d}\xi. \tag{41}$$

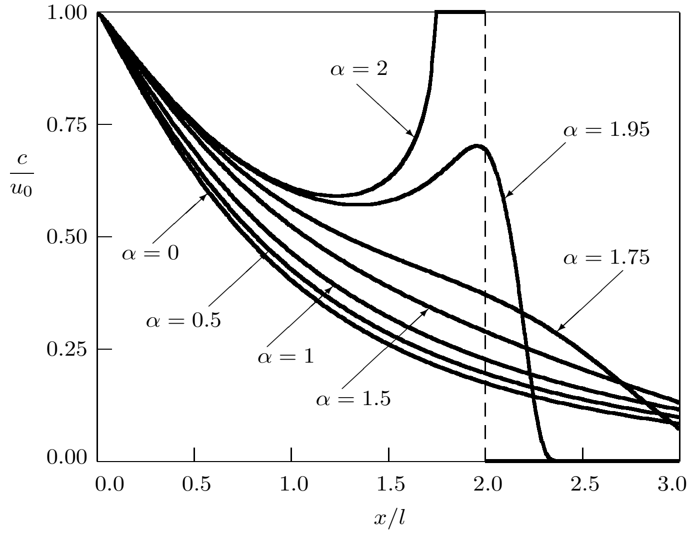

Figure 4: Dependence of solution on distance x/l for $y = 0$ and $\kappa = 2$ (the Dirichlet boundary condition).

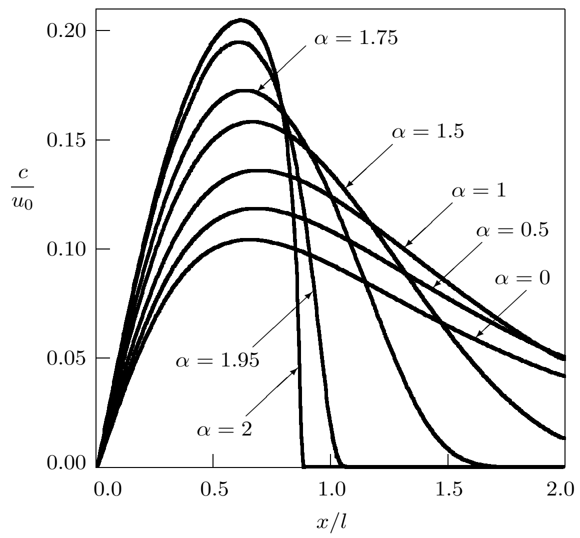

Figure 5: Dependence of solution on distance x/l for $y/l = 1.5$ and $\kappa = 1$ (the Dirichlet boundary condition).

Passing to the polar coordinate we get

$$c = \frac{aW_0 t^{\alpha-1}}{\pi^2} \int_0^\infty \rho E_{\alpha,\alpha}(-a\rho^2 t^\alpha)\, \mathrm{d}\rho \int_{-\pi/2}^{\pi/2} \sin(x\rho\cos\vartheta)\cos(y\rho\sin\vartheta)\, \mathrm{d}\vartheta \tag{42}$$

and

$$\bar{c} = \frac{1}{\pi} \int_0^\infty \sigma\, E_{\alpha,\alpha}(-\sigma^2)\, J_0(\bar{r}\sigma)\, \mathrm{d}\sigma, \tag{43}$$

where Eq. (A3) has been used and the nondimensional quantity $\bar{c}$ has been introduced:

$$\bar{c} = \frac{t}{W_0} c. \tag{44}$$

Let us analize several particular cases.

4.1.1. Classical diffusion ($\alpha = 1$)

In the case of standard diffusion equation using (A10) we obtain (see, for example, [36])

$$\bar{c} = \frac{1}{2\pi} \exp\left(-\frac{\bar{r}^2}{4}\right). \tag{45}$$

4.1.2. Subdiffusion with $\alpha = 1/2$

In this case, using (20) and changing integration with respect to σ and v we arrive at

$$\bar{c} = \frac{1}{2\pi^{3/2}} \int_0^\infty \exp\left(-v^2 - \frac{\bar{r}^2}{8v}\right) \mathrm{d}v. \tag{46}$$

4.1.3. Ballistic diffusion ($\alpha = 2$)

In the case of wave equation

$$E_{2,2}(-\sigma^2) = \frac{\sin\sigma}{\sigma}, \tag{47}$$

and the discontinuous Weber-Schafheitlin type integral (A15) gives

$$\bar{c} = \begin{cases} \dfrac{1}{\pi} \dfrac{1}{\sqrt{1-\bar{r}^2}}, & 0 \le \bar{r} < 1, \\ 0, & 1 < \bar{r} < \infty. \end{cases} \tag{48}$$

Dependence of nondimensional function $\bar{c}$ on nondimensional radial coordinate $\bar{r}$ is shown in Figs. 6 and 7.

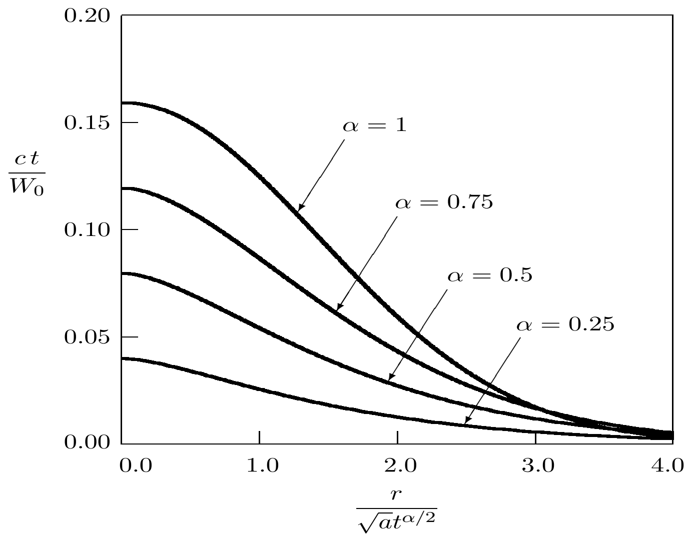

Figure 6: Dependence of solution on distance (the delta pulse boundary condition for the normal derivative; $0 < \alpha \leq 1$).

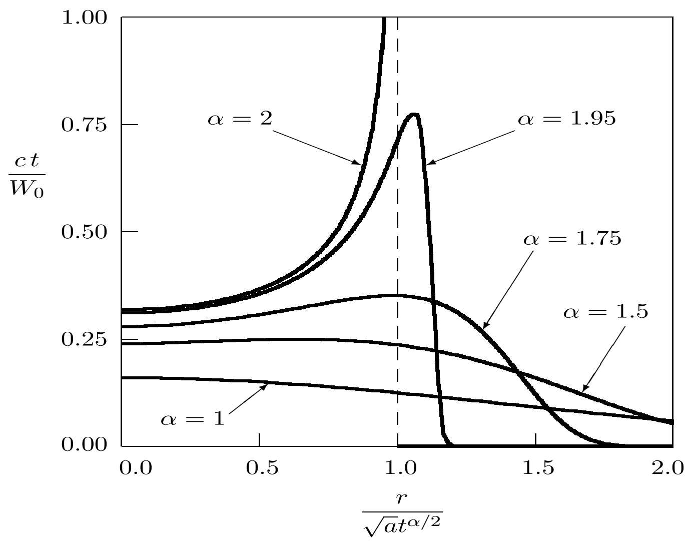

Figure 7: Dependence of solution on distance (the delta pulse boundary condition for the normal derivative; $1 \leq \alpha < 2$).

4.2. The constant boundary value of normal derivative in a local domain

Of particular interest is the initial-boundary value problem (2)–(4) with the constant boundary value of normal derivative of a function in the domain $|y| < l$:

$$x = 0: \quad \frac{\partial c}{\partial x} = \begin{cases} -w_0, & |y| < l, \\ 0, & |y| > l. \end{cases} \tag{49}$$

The integral transforms technique leads to

$$c^* = \frac{2aw_0 \sin(l\eta)}{\sqrt{2\pi}\eta} \frac{1}{s[s^\alpha + a(\xi^2 + \eta^2)]}. \tag{50}$$

Inverting the transforms we obtain

$$c = \frac{2w_0}{\pi^2} \int_{-\infty}^{\infty} \frac{\sin(l\eta)}{\eta} \cos(y\eta)\, \mathrm{d}\eta \times$$

$$\times \int_0^\infty \frac{1}{\xi^2 + \eta^2} \left\{1 - E_\alpha[-a(\xi^2 + \eta^2)t^\alpha]\right\} \cos(x\xi)\, \mathrm{d}\xi \tag{51}$$

or, after passing to the polar coordinates and taking into account integrals (A3) and (A5) from Appendix,

$$\bar{c} = \frac{1}{\pi} \int_0^\infty \left[1 - E_\alpha(-\kappa^2\sigma^2)\right] \frac{1}{\sigma}\, \mathrm{d}\sigma \int_{\bar{y}-1}^{\bar{y}+1} J_0\left(\sigma\sqrt{u^2 + \bar{x}^2}\right) \mathrm{d}u, \tag{52}$$

where

$$\bar{c} = \frac{c}{w_0 l}, \tag{53}$$

and $\bar{x}$, $\bar{y}$ and κ are described by (28).

Examine several particular cases.

4.2.1. Classical diffusion ($\alpha = 1$)

The solution has the form (see also [36])

$$\bar{c} = \frac{1}{\pi} \int_0^\infty \left[1 - \exp(-\kappa^2 \sigma^2)\right] \frac{1}{\sigma} \, \mathrm{d}\sigma \int_{\bar{y}-1}^{\bar{y}+1} J_0\left(\sigma \sqrt{u^2 + \bar{x}^2}\right) \mathrm{d}u. \tag{54}$$

4.2.2. Localized diffusion ($\alpha = 0$)

Using Eq. (A12) we get (see also [36])

$$\bar{c} = \frac{1}{\pi} \int_{\bar{y}-1}^{\bar{y}+1} K_0\left(\frac{\sqrt{u^2 + \bar{x}^2}}{\kappa}\right) \mathrm{d}u. \tag{55}$$

4.2.3. Ballistic diffusion ($\alpha = 2$)

In the case of wave equation, taking Eqs. (A7), (A8) and (A17) into account, we obtain the following solution ($y \geq 0$ is considered as in (36)–(38)):

(i) $0 < \kappa < |1 - \bar{y}|$

$$\bar{c} = \begin{cases} \frac{1}{2}(\kappa - \bar{x})[1 + \mathrm{sign}\,(1 - \bar{y})], & 0 < \bar{x} < \kappa, \\ 0, & \kappa < \bar{x} < \infty; \end{cases} \tag{56}$$

(ii) $|1 - \bar{y}| < \kappa < 1 + \bar{y}$

$$\bar{c} = \begin{cases} \frac{1}{2}(\kappa - \bar{x}) + \frac{\kappa}{\pi} \arcsin \frac{1-\bar{y}}{\sqrt{\kappa^2 - \bar{x}^2}} & \\ -\frac{\bar{x}}{\pi} \arctan \frac{\kappa(1-\bar{y})}{\bar{x}\sqrt{\kappa^2 - \bar{x}^2 - (1-\bar{y})^2}} & \\ +\frac{1}{2\pi}(1 - \bar{y}) \ln \frac{\kappa + \sqrt{\kappa^2 - \bar{x}^2 - (1-\bar{y})^2}}{\kappa - \sqrt{\kappa^2 - \bar{x}^2 - (1-\bar{y})^2}}, & 0 < \bar{x} < \sqrt{\kappa^2 - (1-\bar{y})^2}, \\ \frac{1}{2}(\kappa - \bar{x})[1 + \mathrm{sign}\,(1 - \bar{y})], & \sqrt{\kappa^2 - (1-\bar{y})^2} < \bar{x} < \kappa, \\ 0, & \kappa < \bar{x} < \infty; \end{cases} \tag{57}$$

(iii) $1 + \bar{y} < \kappa < \infty$

$$\bar{c} = \begin{cases} \frac{1}{\pi}\Big[\kappa \arcsin \frac{1-\bar{y}}{\sqrt{\kappa^2-\bar{x}^2}} \\ +\kappa \arcsin \frac{1+\bar{y}}{\sqrt{\kappa^2-\bar{x}^2}} \\ -\bar{x} \arctan \frac{\kappa(1-\bar{y})}{\bar{x}\sqrt{\kappa^2-\bar{x}^2-(1-\bar{y})^2}} \\ -\bar{x} \arctan \frac{\kappa(1+\bar{y})}{\bar{x}\sqrt{\kappa^2-\bar{x}^2-(1+\bar{y})^2}} \\ +\frac{1}{2}(1-\bar{y}) \ln \frac{\kappa+\sqrt{\kappa^2-\bar{x}^2-(1-\bar{y})^2}}{\kappa-\sqrt{\kappa^2-\bar{x}^2-(1-\bar{y})^2}} \\ +\frac{1}{2}(1+\bar{y}) \ln \frac{\kappa+\sqrt{\kappa^2-\bar{x}^2-(1+\bar{y})^2}}{\kappa-\sqrt{\kappa^2-\bar{x}^2-(1+\bar{y})^2}}\Big], & 0<\bar{x}<\sqrt{\kappa^2-(1+\bar{y})^2}, \\ \frac{1}{2}(\kappa-\bar{x}) + \frac{\kappa}{\pi} \arcsin \frac{1-\bar{y}}{\sqrt{\kappa^2-\bar{x}^2}} \\ -\frac{\bar{x}}{\pi} \arctan \frac{\kappa(1-\bar{y})}{\bar{x}\sqrt{\kappa^2-\bar{x}^2-(1-\bar{y})^2}} \\ +\frac{1}{2\pi}(1-\bar{y}) \ln \frac{\kappa+\sqrt{\kappa^2-\bar{x}^2-(1-\bar{y})^2}}{\kappa-\sqrt{\kappa^2-\bar{x}^2-(1-\bar{y})^2}}, & \sqrt{\kappa^2-(1+\bar{y})^2}<\bar{x}<\sqrt{\kappa^2-(1-\bar{y})^2}, \\ \frac{1}{2}(\kappa-\bar{x})[1+\operatorname{sign}(1-\bar{y})], & \sqrt{\kappa^2-(1-\bar{y})^2}<\bar{x}<\kappa, \\ 0, & \kappa<\bar{x}<\infty. \end{cases} \tag{58}$$

Dependence of nondimensional solution $\bar{c}$ on nondimensional distance x/l is shown in Fig. 8 for $y = 0$ and $\kappa = 1$. The plots of $\bar{c}$ versus y/l at the boundary $x = 0$ are depicted in Fig. 9 for $\kappa = 1.5$.

5. Concluding remarks

The results given by Eqs. (16), (27), (43) and (52) and displayed in Figs. 1–9 are the primary results of this paper. The solutions of time-fractional diffusion equation satisfy the appropriate boundary conditions at the boundary $x = 0$ and reduce to the solutions of classical diffusion equation in the limit $\alpha = 1$. In the case $0 < \alpha < 1$ the time-fractional diffusion equation interpolates the Helmholtz equation and diffusion equation. In this case, when it is possible to consider the limit $\alpha \to 0$, the obtained solutions reduce to the solutions of Helmholtz equation. In the case $1 < \alpha < 2$ the time-fractional diffusion equation interpolates the diffusion equation

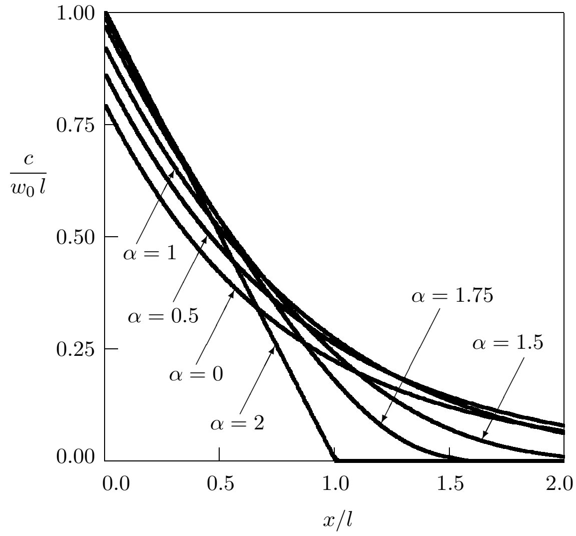

Figure 8: Dependence of solution on distance x/l for $y = 0$ and $\kappa = 1$ (the Neumann boundary condition).

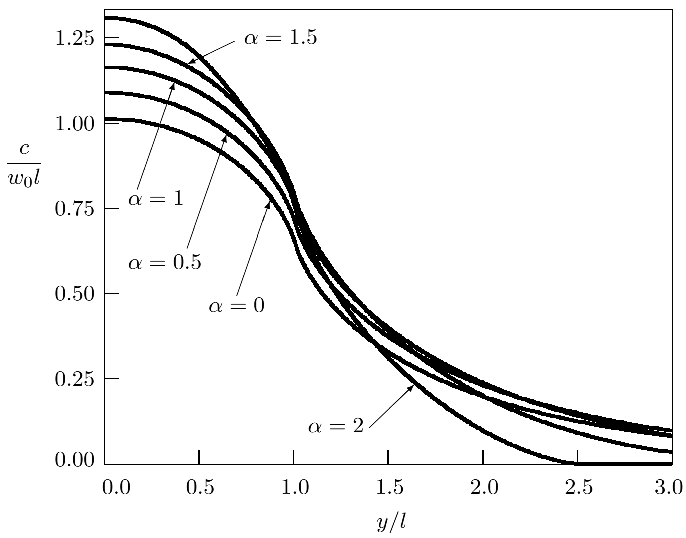

Figure 9: Dependence of solution on distance y/l for $x = 0$ and $\kappa = 1.5$ (the Neumann boundary condition).

and wave equation. In the limit $\alpha = 2$ the obtained solutions reduce to the solutions of wave equation. The solutions of fractional diffusion equation in the case $1 < \alpha < 2$ approximate propagating steps and humps typical for the standard wave equation in contrast to the shape of curves describing the slow diffusion regime. In particular, it is evident from Figures how wave fronts arising in the case of the wave equation are approximated by the solutions of time-fractional diffusion equation with α approaching the value 2. As the numerical values of $\bar{c}$ for $0 < \alpha \le 1$ and $1 \le \alpha < 2$ in the case of delta pulses boundary conditions are widely different, the typical results for $0 < \alpha \le 1$ and $1 \le \alpha < 2$ are presented in Figs. 1 and 2 for the Dirichlet boundary condition and in Figs. 6 and 7 for the Neumann boundary condition, respectively, using different scales.

Appendix

Here we present some integrals used in the paper.

Integrals containing elementary functions are borrowed from [38]:

$$\int_0^\infty \frac{x}{x^2+p^2} \sin qx \, \mathrm{d}x = \frac{\pi}{2} \mathrm{e}^{-pq}, \qquad p > 0, \quad q > 0, \tag{A1}$$

$$\int_0^\infty \frac{\mathrm{e}^{-px}}{x} \sin qx \, \mathrm{d}x = \arctan \frac{q}{p}, \qquad p > 0, \tag{A2}$$

$$\int_0^1 \frac{\cos(p\sqrt{1-x^2})}{\sqrt{1-x^2}} \cos qx \, \mathrm{d}x = \frac{\pi}{2} J_0\left(\sqrt{p^2+q^2}\right), \tag{A3}$$

$$\int_0^1 \sin(p\sqrt{1-x^2}) \cos qx \, \mathrm{d}x = \frac{\pi}{2} \frac{p}{\sqrt{p^2+q^2}} J_1\left(\sqrt{p^2+q^2}\right). \tag{A4}$$

Integrating both sides of (A3) and (A4) with respect to q allows us to obtain the additional integrals

$$\int_0^1 \frac{\cos(p\sqrt{1-x^2})}{x\sqrt{1-x^2}} \sin qx \, \mathrm{d}x = \frac{\pi}{2} \int_0^q J_0\left(\sqrt{p^2+u^2}\right) \mathrm{d}u, \qquad q > 0, \tag{A5}$$

$$\int_0^1 \frac{\sin(p\sqrt{1-x^2})}{x} \sin qx \, \mathrm{d}x = \frac{\pi}{2} \int_0^q \frac{p}{\sqrt{p^2+u^2}} J_1\left(\sqrt{p^2+u^2}\right) \mathrm{d}u, \quad q > 0. \tag{A6}$$

We also need the following integrals from [39], [40]:

$$\int \frac{1}{1+\varepsilon\cos x}\,\mathrm{d}x = \frac{2}{\sqrt{1-\varepsilon^2}}\arctan\left(\sqrt{\frac{1-\varepsilon}{1+\varepsilon}}\tan\frac{x}{2}\right), \quad 0<\varepsilon<1, \tag{A7}$$

$$\int \frac{1}{p^2-q^2\cos^2 x}\,\mathrm{d}x = \frac{1}{p^2\sin\gamma}\,\arctan\left(\frac{\tan x}{\sin\gamma}\right), \tag{A8}$$

where $q^2 < p^2$, $\gamma = \arccos\frac{q}{p}$.

Integrals containing Bessel functions of the first kind, taken from [41]:

$$\int_0^\infty \mathrm{e}^{-px^2}\,J_1(qx)\,\mathrm{d}x = \frac{1}{q}\left[1-\exp\left(-\frac{q^2}{4p}\right)\right], \qquad p>0, \quad q>0, \tag{A9}$$

$$\int_0^\infty x\,\mathrm{e}^{-px^2}\,J_0(qx)\,\mathrm{d}x = \frac{1}{2p}\,\exp\left(-\frac{q^2}{4p}\right), \qquad p>0, \quad q>0, \tag{A10}$$

$$\int_0^\infty x^2\mathrm{e}^{-px^2}\,J_1(qx)\,\mathrm{d}x = \frac{q}{4p^2}\,\exp\left(-\frac{q^2}{4p}\right), \qquad p>0, \quad q>0, \tag{A11}$$

$$\int_0^\infty \frac{x}{x^2+p^2}\,J_0(qx)\,\mathrm{d}x = K_0(pq), \qquad p>0, \quad q>0, \tag{A12}$$

$$\int_0^\infty \frac{x^2}{x^2+p^2}\,J_1(qx)\,\mathrm{d}x = pK_1(pq), \qquad p>0, \quad q>0, \tag{A13}$$

where $K_n(x)$ are the modified Bessel functions of order n.

Eq. (A13) can be used to get the following integral

$$\int_0^\infty \frac{1}{x^2+p^2}\,J_1(qx)\,\mathrm{d}x = \frac{1}{qp^2} - \frac{1}{p}K_1(pq), \qquad p>0, \quad q>0. \tag{A14}$$

The discontinuous Weber–Schafheitlin type integrals appear in the case of the wave equation and read [41]

$$\int_0^\infty \sin px\,J_0(qx)\,\mathrm{d}x = \begin{cases} \dfrac{1}{\sqrt{p^2-q^2}}, & 0<q<p, \\ 0, & 0<p<q, \end{cases} \tag{A15}$$

$$\int_0^\infty \cos px\,J_1(qx)\,\mathrm{d}x = \begin{cases} -\dfrac{q}{\sqrt{p^2-q^2}}\,\dfrac{1}{p+\sqrt{p^2-q^2}}, & 0<q<p, \\ q^{-1}, & 0<p<q, \end{cases} \tag{A16}$$

$$\int_0^\infty \frac{1-\cos px}{x} J_0(qx)\,\mathrm{d}x = \begin{cases} \ln \dfrac{p+\sqrt{p^2-q^2}}{q}, & 0<q<p, \\ 0, & 0<p<q. \end{cases} \qquad \text{(A17)}$$

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